The Efficiency of the Semi-Direct Products of Free Abelian Monoid with Rank $n$ by the Infinite Cyclic Monoid

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Abstract. In this paper we give necessary and sufficient conditions for the efficiency of the semi-direct product of free abelian monoid with rank $n$ by the infinite cyclic monoid.

Keywords: Efficiency, Semi-direct product, Monoid.


INTRODUCTION

Let $\mathcal{P} = [x;r]$ be a finite presentation for a monoid $M$. Then the Euler characteristic of $\mathcal{P}$ is defined by $\chi(\mathcal{P}) = 1 - |x| + |r|$ and an upper bound of $M$ is defined by $\delta(M) = 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M))$. In an unpublished work, S.J. Pride has shown that $\chi(\mathcal{P}) \geq \delta(M)$. With this background, one can define a monoid presentation $\mathcal{P}$ to be efficient if $\chi(\mathcal{P}) = \delta(M)$, and then $M$ is called efficient if it has an efficient presentation.

It is well known that one of the effective ways to show efficiency for the monoid $M$ is to use spherical monoid pictures over $\mathcal{P}$. These geometric configurations are the representative elements of the Squier complex denoted by $\mathcal{P}(\mathcal{P})$ (see, for example [4], [5], [7]). Suppose $Y$ is a collection of spherical monoid pictures over $\mathcal{P}$. Two monoid pictures $P$ and $P'$ are equivalent relative to $Y$ if there is a finite sequence of monoid pictures $P = P_0, P_1, \ldots, P_m = P'$ where, for $1 \leq i \leq m$, the monoid picture $P_i$ is obtained from the picture $P_{i-1}$ either by the insertion, deletion and replacement operations. By definition, a set $Y$ of spherical monoid pictures over $\mathcal{P}$ is a trivializer of $\mathcal{P}(\mathcal{P})$ if every spherical monoid picture is equivalent to an empty picture relative to $Y$. The trivializer is also called a set of generating pictures.

For any monoid picture $P$ over $\mathcal{P}$ and for any $R \in r$, exp$_r(P)$ denotes the exponent sum of $R$ in $P$ which is the number of positive discs labelled by $R$, minus the number of negative discs labelled by $R$. For a non-negative integer $n$, $\mathcal{P}$ is said to be $n$-Cockcroft if exp$_r(P) \equiv 0 \mod n$, (where congruence $\mod n$ is taken to be equality) for all $R \in r$ and for all spherical pictures $P$ over $\mathcal{P}$. Then a monoid $M$ is said to be $n$-Cockcroft if it admits an $n$-Cockcroft presentation. In fact to verify the $n$-Cockcroft property, it is enough to check for pictures $P \in Y$, where $Y$ is a trivializer (see [4], [5]). The 0-Cockcroft property is usually just called Cockcroft.

The following result is also an unpublished result by S.J. Pride.

Theorem 1 Let $\mathcal{P}$ be a monoid presentation. Then $\mathcal{P}$ is efficient if and only if it is $p$-Cockcroft for some prime $p$.

Let $K$ be free abelian monoid of rank $n$ with $\mathcal{P}_K = \{y_1, y_2, \ldots, y_n; y_iy_j = y_jy_i \ (1 \leq i < j \leq n)\}$, and let $A$ be the infinite cyclic monoid with $\mathcal{P}_A = [x;]$. Also let $\psi_A$ be an endomorphism of $K$ where $A$ is the matrix on the positive integer

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the presentation

\[ \mathcal{M} = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{bmatrix}. \]

given by \( y_i \mapsto y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \cdots y_n^{\alpha_{in}} \) for \( 1 \leq i \leq n \). Thus we have the presentation of \( M = K \rtimes_{\theta} A \) where \( \theta : A \rightarrow \text{End}(K) \), \( x \mapsto \Psi_x \).

\[ \mathcal{P}_M = \{ y_1 y_2, \ldots, y_n x_1, y_j y_i = y_i y_j \ (1 \leq i < j \leq n), T_{y_k} (1 \leq i \leq n) \}, \]

for the monoid \( M \) where

\[ T_{y_k} ; y^n_k = y_1^{\alpha_{k1}} y_2^{\alpha_{k2}} \cdots y_n^{\alpha_{kn}}. \]

Now, by considering the presentation \( \mathcal{P}_M \) in (1), we prove the following theorem as a main result in the present paper.

**Theorem 2** Let \( p \) be a prime or 0. Then the presentation \( \mathcal{P}_M \) is \( p \)-Cockcroft if and only if, for all \( 1 \leq i < j \leq n \) and \( 1 \leq k, m \leq n \),

\[ \alpha_{jk} \alpha_{k+m} - \alpha_k \alpha_{jk+m} \equiv \begin{cases} 1 \pmod{p}, & \text{if } k = i \text{ and } k + m = j, \\ 0 \pmod{p}, & \text{otherwise}. \end{cases} \]

We may refer [1, 2, 3, 4, 5, 6, 7] to the reader for most of the fundamental material (for instance, semidirect products of monoids, Squier complex, a trivializer set of the Squier complex, spherical and non-spherical monoid pictures) which will be needed here.

**TRIVIALIZER SET** \( \mathcal{R}(\mathcal{P}_M) \)

Let us consider the relation \([yz]_{\Psi_x} \in \mathcal{P}_K \) where \( (yz)_{\Psi_x} \in \mathcal{P}_K \). Because of this, we get a non-spherical picture, say \( B_{S,x} \), over \( \mathcal{P}_K \) where \( S : y_i y_j = y_j y_i \ (1 \leq i < j \leq n) \). Thus by using the subpicture \( B_{S,x}, T_{y_k} \) discs and \( S \) disc, we have the generating pictures, say \( P_{S,x} \). Also, let \( C \) consists of the pictures \( P_{S,x} \). Let \( X_K \) be trivializer set of \( \mathcal{R}(\mathcal{P}_K) \). We should note that since the monoid \( A \) is the infinite cyclic monoid, we don’t have a trivializer set of \( \mathcal{R}(\mathcal{P}_A) \). Let us consider the presentation \( \mathcal{P}_M \), as in (1). Then, by [8], a trivializer set of \( \mathcal{R}(\mathcal{P}_M) \) is

\[ X_K \cup C. \]

The reason for us keeping work on the above monoid pictures is their usage in the important connection between efficiency and \( p \)-Cockcroft property. Therefore, in the present paper, we will use this connection to get the efficiency. To do that we will count the exponent sums of the discs in these above pictures to obtain \( p \)-Cockcroft property for the presentation \( \mathcal{P}_M \) given in (1).

**PROOF OF THE MAIN RESULT AND ITS APPLICATION**

Let us consider the discs given in \( P_{S,x} \). To prove Theorem 2, we will count the exponent sums of the discs in these pictures. Here, when we calculate the number of \( S \) discs in \( B_{S,x} \) where \( S : y_i y_j = y_j y_i \) such that \( i < j \) and \( i, j \in \{1, 2, \ldots, n\} \), we see that it is equal to \( \alpha_{ij} \alpha_{ij} - \alpha_{ij} \alpha_{ij} \). On the other hand, we have also \( y_1 y_2 = y_2 y_1, y_1 y_3 = y_3 y_1, \ldots, y_{n-1} y_n = y_n y_{n-1} \) discs different from \( S : y_i y_j = y_j y_i \) in \( B_{S,x} \), say \( \hat{S} \) discs. The number of \( \hat{S} \) discs is \( \alpha_{k+m} - \alpha_k \alpha_{m+k} \) where \( 1 \leq k, m \leq n \). At this point, it is easy to see that

\[ \text{exp}_S(P_{S,x}) = 1 - \text{exp}_S(B_{S,x}), \]

and to \( p \)-Cockcroft property be hold, we need to have

\[ \text{exp}_S(P_{S,x}) \equiv 0 \pmod{p} \iff \text{exp}_S(B_{S,x}) \equiv 1 \pmod{p}, \]

\[ \text{exp}_P(P_{S,x}) \equiv 0 \pmod{p} \iff \text{exp}_P(B_{S,x}) \equiv 0 \pmod{p}. \]
By using this, if \( k = i \) and \( j = k + m \), we have \( \alpha_{jk} \alpha_{ik+m} - \alpha_{ik} \alpha_{jk+m} \equiv 1 \mod p \). Otherwise, we get that \( \alpha_{jk} \alpha_{ik+m} - \alpha_{ik} \alpha_{jk+m} \equiv 0 \mod p \). Moreover, in \( P_{S,x} \), we also have 2 times positive and 2 times negative \( T_{y_i x} \) discs. That means
\[
expr_{y_i x}(P_{S,x}) = 0,
\]
and so we say that \( p \)-Cockcroft property is hold for these discs. Hence the result.

We note that, by considering the trivializer set \( X_K \) of the Squier complex \( \mathcal{D}(\mathcal{P}_K) \), it can be easily deduced that \( \mathcal{P}_K \) are \( p \)-Cockcroft, in fact Cockcroft, presentations.

These all above procedure give us sufficient conditions to be the presentation \( \mathcal{P}_M \) in (1) is \( p \)-Cockcroft for any prime \( p \). In fact the converse part (necessary conditions) of the theorem is quite clear.

Let us suppose that the monoid \( K \) is presented by \( \mathcal{P}_K = [y_1, y_2 : y_1 y_2 = y_2 y_1] \). Hence we get the corresponding semi-direct product \( M \) with the presentation
\[
\mathcal{P}_M = [y_1, y_2, x : y_1 y_2 = y_2 y_1, y_1 x = x y_1^{\alpha_1}, y_2 x = x y_2^{\alpha_2}].
\]
(2)

Let us consider presentation given in (2). Then we can give the following corollary as a consequence of the main result.

**Corollary 3** Let \( p \) be a prime or 0. Then the presentation \( \mathcal{P}_M \) is \( p \)-Cockcroft if and only if
\[
\alpha_{21} \alpha_{12} - \alpha_{11} \alpha_{22} \equiv 1 \mod p.
\]

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