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Citation: AIP Conf. Proc. 1470, 50 (2012); doi: 10.1063/1.4747636
View online: http://dx.doi.org/10.1063/1.4747636
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1470&Issue=1
Published by the American Institute of Physics.

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The Nature of Solutions of The Difference Equation

\[ x_n = \max \left\{ \frac{A}{x_{n-2}}, \frac{B}{x_{n-3}^{\alpha}} \right\} \]

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Abstract. We consider positive solutions of the difference equation

\[ x_n = \max \left\{ \frac{A}{x_{n-2}}, \frac{B}{x_{n-3}^{\alpha}} \right\}, \quad n \geq 0, \]

where \( A \geq 0, B \geq 0, 0 < \alpha \leq 1 \) and the initial conditions \( x_{-1}, x_{-2}, x_{-3} \) are arbitrary positive real numbers. We show that every positive solution of this difference equation approaches \( x = \sqrt{B} \) or is eventually periodic with period 4, 5 or 6. Also, this work confirms partially the conjecture proposed in [18].

Keywords: Max operator, Difference equation, Positive solution, Stability, Periodicity.

PACS: 02.70.Bf, 02.30.Jr

INTRODUCTION

In this paper, we investigate the asymptotic behavior and periodic nature of solutions of the following difference equation

\[ x_n = \max \left\{ \frac{A}{x_{n-2}}, \frac{B}{x_{n-3}^{\alpha}} \right\}, \quad n \geq 0, \] (1)

where \( A \geq 0, B \geq 0, 0 < \alpha \leq 1 \) and the initial conditions \( x_{-1}, x_{-2}, x_{-3} \) are arbitrary positive real numbers. We prove that every positive solution of this difference equation approaches \( x = \sqrt{B} \) or is eventually periodic with period 4, 5 or 6. Also, this paper confirms the conjecture proposed in [18] partially.

The Eq. (1) is motivated by the paper [18]. In [18], it was studied the asymptotic behavior of positive solutions of the difference equation

\[ x_n = \max \left\{ \frac{1}{x_{n-1}^{\alpha_1}}, \frac{A}{x_{n-2}} \right\}, \quad n \geq 0, \] (2)

where \( 0 < \alpha < 1 \) and \( A > 0 \). It was shown that every positive solution of this difference equation approaches \( x = 1 \) or is eventually periodic with period 4. Also, the authors of [18] proposed the following difference equation and conjecture.

\[ x_n = \max \left\{ \frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \ldots, \frac{A_p}{x_{n-p}} \right\}, \quad n \geq 0, \] (3)

where \( A_1 \geq 0, A_2 \geq 0, \ldots, A_p \geq 0, 0 < \alpha_1 \leq 1, \ 0 < \alpha_2 \leq 1, \ldots, 0 < \alpha_p \leq 1, \ \max \{ \alpha_1, \alpha_2, \ldots, \alpha_p \} = 1. \)

**Conjecture 1.** Let \( \alpha_q = 1 \). If \( A_q > \max \{ A_j : 1 \leq j \leq p, j \neq q \} \), then every positive solution of the Eq. (3) is eventually periodic with period \( T=2q \).
PRELIMINARIES

In [15], the following difference equation was investigated

\[ x_{n+1} = \max \left\{ \frac{A}{x_{n-k}}, \frac{B}{x_{n-m}} \right\}, n = 0, 1, \ldots, \]  

(4)

where \( A, B \) are any positive real numbers and \( m, k \in \mathbb{N} \). The following result was given.

**Theorem 1.** Let \( A, B \in (0, \infty) \) and \( m, k \in \mathbb{N} \). Then there exists a positive integer \( T \) such that every positive solution \( \{x_n\} \) of the Eq. (4) is eventually periodic with period \( T \).

In addition, the period \( T \) determined as follows:

\[ T = 2k \text{ if either } A > B \text{ or } A = B \text{ and } m = 3k, \]
\[ T = 2m \text{ if either } A < B \text{ or } A = B \text{ and } k = 3m, \]
\[ T = k + m \text{ if } A = B \text{ and neither } k = 3m \text{ nor } m = 3k. \]

It has been seen easily that for \( k = 2 \) and \( m = 3 \), the Eq. (4) corresponds to the Eq. (1) with \( \alpha = 1 \).

It is easy to see that the solutions of the Eq. (1) are \( x_n = 0, x_n = \frac{B}{x_{n-2}} \) and \( x_n = \frac{A}{x_{n-3}} \) respectively, in the cases \( A = B = 0, A = 0 \) and \( B \neq 0 \) and \( A \neq 0 \) and \( B = 0 \).

**THE CASE A = B**

Let \( x_n = \sqrt{A} C^n \) and \( 0 < C < 1 \) for \( n \geq -3 \). This substitution transforms the Eq. (1) into the difference equation

\[ y_n = \min \left\{ -y_{n-2}, -\alpha y_{n-3} \right\}, \quad n \geq 0, \]

(5)

where \( 0 < \alpha < 1 \).

**Lemma 1.** Let \( \{y_n\} \) be a solution of the Eq. (5). Then,

\[ |y_n| \leq \max \{|y_{n-2}|, \alpha |y_{n-3}|\} \]

(6)

for all \( n \geq 0 \).

**Theorem 2.** If \( \{x_n\} \) is a positive solution of the Eq. (1), then \( \{x_n\} \) approaches \( \bar{x} = \sqrt{B} \).

**THE CASE A < B**

We consider the Eq. (1), where \( 0 < \alpha < 1 \) and \( 0 < A < B \). Let \( x_n = \sqrt{B} c^n, n \geq -3 \). Then, we have the difference equation

\[ z_n = \min \left\{ 1 - z_{n-2}, -\alpha z_{n-3} \right\}, \quad n \geq 0, \]

(7)

where the initial conditions \( z_{-1}, z_{-2}, z_{-3} \) are real numbers and \( C = \frac{A}{B} \).

**Lemma 2.** Let \( \{z_n\} \) be a solution of Eq. (7). Then,

\[ |z_n| \leq \max \{|z_{n-2}| - 1, \alpha |z_{n-3}|\} \]

(8)

for all \( n \geq 0 \).

**Theorem 3.** If \( \{x_n\} \) is a positive solution of the Eq. (1), then \( \{x_n\} \) approaches \( \bar{x} = \sqrt{B} \).

**THE CASE A > B**

The substitution \( x_n = \sqrt{B} C^n, n \geq -3 \), transforms the Eq. (1) to the following difference equation

\[ w_n = \max \left\{ 1 - w_{n-2}, -\alpha w_{n-3} \right\}, \quad n \geq 0, \]

(9)

where the initial conditions are real numbers and \( C = \frac{A}{B} \).

**Theorem 4.** Every positive solution of the Eq. (1) in the case \( A > B \) is eventually periodic with period 4.
REFERENCES